THEORY OF INTEREST AND LIFE CONTINGENCIES WITH PENSION APPLICATIONS

A Problem-Solving Approach

Third Edition

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To my Mother and the memory of my Father

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PREFACE

It is impossible to escape the practical implications of compound interest in our modern society. The consumer is faced with a bewildering choice of bank accounts offering various rates of interest, and wishes to choose the one which will give the best return on her savings. A home-buyer is offered various mortgage plans by different companies, and wishes to choose the one most advantageous to him. An investor seeks to purchase a bond which pays coupons on a regular basis and is redeemable at some future date; again, there are a wide variety of choices available.

Comparing possibilities becomes even more difficult when the payments involved are dependent on the individual's survival. For example, an employee is offered a variety of different pension plans and must decide which one to choose. Also, most people purchase life insurance at some point in their lives, and a bewildering number of different plans are offered.

The informed consumer must be able to make an intelligent choice in situations like those described above. In addition, it is important that, whenever possible, she be able to make the appropriate calculations herself in such cases. For example, she should understand why a given series of mortgage payments will, in fact, pay off a certain loan over a certain period of time. She should also be able to decide which portion of a given payment is paying off the balance of the loan, and which portion is simply paying interest on the outstanding loan balance.

The first goal of this text is to give the reader enough information so that he can make an intelligent choice between options in a financial situation, and can verify that bank balances, loan payments, bond coupons, etc. are correct. Too few people in today's society understand how these calculations are carried out.

In addition, however, we are concerned that the student, besides being able to carry out these calculations, understands why they work. It is not enough to memorize a formula and learn how to apply it; you should understand why the formula is correct. We also wish to present the material in a proper mathematical setting, so the student will see how the theory of interest is interrelated with other branches of mathematics.

Let me explain why the phrase "Problem-Solving Approach" appears in the title of this text. We will prove a very small number of formulae and then concentrate our attention on showing how these formulae can be applied to a wide variety of problems. Skill will be needed to take the data presented in a particular problem and see how to rearrange it so the formulae can be used. This approach differs from many texts, where a large number of formulae are presented, and the student tries to memorize which problems can be solved by direct application of a particular formula. We wish to emphasize understanding, not rote memorization.

A working knowledge of elementary calculus is essential for a thorough understanding of all the material. However, a large portion of this book can be read by those without such a background by omitting the sections dependent on calculus. Other required background material such as geometric sequences, probability and expectation, is reviewed when it is required.

Each chapter in this text includes a large number of examples and exercises. It should be obvious that the most efficient way for a student to learn the material is for her to work all the exercises.

Finally, let us stress that it is assumed that every student has a calculator (with a y^x button) and knows how to use it. It is because of our ability to use a calculator that many formulae mentioned in older texts on the subject are now unnecessary.

This book is naturally divided into two parts. Chapters 1-5 are concerned solely with the Theory of Interest, and Life Contingencies is introduced in Chapters 6-11.

In Chapter 1 we present the basic theory concerning the study of interest. Our goal here is to give a mathematical background for this area, and to develop the basic formulae which will be needed in the rest of the book. Students with a weak calculus background may wish to omit Section 1.6 on the force of interest, as it is of more theoretical than practical importance. In Chapter 2 we show how the theory in Chapter 1 can be applied to practical problems. The important concept of equation of value is introduced, and many worked examples of numerical problems are presented. Chapter 3 discusses the extremely important concept of annuities. After developing a few basic formulae, our main emphasis in this chapter is on practical problems, seeing how data for such problems can be substituted in the basic formulae. It is in this section especially that we have left out many of the formulae presented in other texts, preferring to concentrate on problem-solving techniques rather than rote memorization. Chapters 4 and 5 deal with further applications of the material in Chapters 1 through 3, namely amortization, sinking funds and bonds.

Chapter 6 begins with a review of the important concepts of probability and expectation, and then illustrates how probability can be combined with the theory of interest. In Chapter 7 we introduce life tables, discussing how they are constructed and how they can be applied. Chapter 8 is concerned with life annuities, that is annuities whose payment are conditional on survival, and Chapter 9 discusses life insurance. These ideas are generalized to multi-life situations in Chapter 10.

Finally, Chapter 11 demonstrates how many of these concepts are applied in the extremely important area of pension plans.

Chapters 1 through 6 have been used for several years as the text material for a one semester undergraduate course in the Theory of Interest, and I would like to thank those students who pointed out errors in earlier drafts. In addition, I am deeply indebted to Brenda Crewe and Wanda Heath for an excellent job of typing the manuscript, and to my colleague, Dr. P. P. Narayanaswami, for his invaluable technical assistance.

Chuck Vinsonhaler, University of Connecticut, was strongly supportive of this project, and introduced me to the people at ACTEX Publications, for which I owe him a great deal. Dick London did the technical content editing, Marilyn Baleshiski provided the electronic typesetting, and Marlene Lundbeck designed the text's cover. I would like to thank them for taking such care in turning a very rough manuscript into what I hope is a reasonably comprehensive yet friendly and readable text book for actuarial students.

St. John’s, Newfoundland Michael M. Parmenter
December, 1988

PREFACE TO THE REVISED EDITION

In the fifteen months since the original edition of this text was published, a number of comments have been received from teachers and students regarding that edition.

We are pleased to note that most of the comments have been quite complimentary to the text, and we are making no substantial modifications at this time.

A significant, and thoroughly justified, criticism of the original edition is that time diagrams were not used to illustrate the examples given in the second half of the text, and that deficiency has been rectified by the inclusion of thirty-five additional figures in the Revised Edition.

Thus it is fair to say that there are no *new topics* contained in the Revised Edition, but rather that the pedagogy has been strengthened. For this reason we prefer to call the new printing a Revised Edition, rather than a Second Edition.

In addition we have corrected the errata in the original edition. We would like to thank all those who took the time to bring the various errata to our attention.

February, 1990 M.M.P.

PREFACE TO THE THIRD EDITION

It is now more than ten years since the original publication of this textbook. In that time, several very significant developments have occurred to suggest that a new edition of the text is now needed, and those developments are reflected in the modifications and additions made in this Third Edition.

First, improvements in calculator technology give us better approaches to reach numerical results. In particular, many calculators now include iteration algorithms to permit direct calculation of unknown annuity interest rates and bond yield rates. Accordingly, the older approximate methods using interpolation have been deleted from the text.

Second, with the discontinued publication of the classic textbook *Life Contingencies* by C.W. Jordan, our text has become the only one published in North America which provides the traditional presentation of contingency theory. To serve the needs of those who still prefer this traditional approach, including the use of commutation functions and a deterministic life table model, we have chosen to include various topics contained in Jordan's text but not contained in our earlier editions. These include insurances payable at the moment of death (Section 9.3), life contingent accumulation functions (Section 8.2), the table of uniform seniority concept for use with Makeham and Gompertz annuity values (Section 11.1), simple contingent insurance functions (Section 11.1), and an expansion of the material regarding multiple-decrement theory (Section 7.6).

Third, actuaries today are interested in various concepts of finance beyond those included in traditional interest theory. To that end we have introduced the ideas of real rates of return, investment duration, modified duration, and so on, in this Third Edition.

Fourth, the new edition provides a gentle introduction to the more modern stochastic view of contingency theory, in the completely new Chapter 10, to supplement the traditional presentation.

In connection with the expansion of topics, the new edition contains over forty additional exercises and examples. As well, the numerical answers to the exercises have been made more precise and the errata in the previous edition have been corrected. We would like to thank everyone who brought such errata to our attention.

With the considerable modifications made in the new edition, we believe this text is now appropriate for two major audiences: pension actuaries, who wish to understand the use of commutation functions and deterministic contingency theory in pension mathematics, and university students, who seek to understand basic contingency theory at an introductory level before undertaking a study of the more mathematically sophisticated stochastic contingency theory.

As with the original edition of this text, the staff at ACTEX Publications has been invaluable in the development of this new edition. Specifically I would like to thank Denise Rosengrant for her text composition and typesetting work, and Dick London, FSA, for his technical content editing.

February, 1999 M.M.P

CHAPTER TWO
INTEREST: BASIC APPLICATIONS

2.1 EQUATION OF VALUE

In its simplest terms, every interest problem involves only four quantities: the principal originally invested, the accumulated value at the end of the period of investment, the period of investment, and the rate of interest. Any one of these four quantities can be calculated if the others are known.

In this section we will present a number of examples illustrating the determination of principal, accumulated value, and period of investment; determining the rate of interest will be explored in Sections 2.2 and 2.3. More complicated situations involving several "principals" invested at different times will arise in practice, and we will examine some of these as well.

The most important tool in dealing with such problems is the time diagram, which we encountered in chapter one, and the first step in any solution should be to draw such a diagram. After that, all entries on the diagram should be "brought" to the same point in time, in order that they can be compared. Then an *equation of value* is set up at that point in time, and a solution is obtained by algebraic means. The student should carefully study the examples in this section to see how these steps are carried out in practice.

We remark that before calculators came into general use, the calculations involved in some of these problems were quite difficult, and it was necessary to develop a collection of techniques to deal with them. Interest tables and log tables were in frequent use, and values which did not appear in the interest tables were handled by interpolation or other approximate methods. For example, the power series expansions given in the previous chapter could be used for calculation, since the first few terms often give a good approximation to the correct answer. We, however, will use our calculators freely and will generally not need to employ the older techniques. That does not mean that every question can be solved by pushing the appropriate button, however; in particular we will see cases where some approximate method (e.g., linear interpolation) is required to obtain an answer. In addition it is often necessary to first analyze the data very carefully, and organize it in such a way such that the calculator can then be called upon to assist in solving the problem. After all, your calculator is only an aid to mechanical computation. The person with the problem still has to solve it!

Example 2.1

Find the accumulated value of 500 after 173 months at a rate of interest of 14% convertible quarterly, assuming compound interest throughout.

Solution

The effective rate of interest is .035 per 3 month period, and there are a total of $57\frac{2}{3}$ periods. Hence the answer is $500(1.035)^{173/3} = 3635.22$. □

Remarks

1. It is quite common to assume compound interest over integral durations, but simple interest between integral durations. Under that assumption, the answer to this example would be $500(1.035)^{57}\left[1+(.035)\left(\frac{2}{3}\right)\right] = 3635.69$. Observe that this answer is larger than the one in the example, agreeing with our earlier observation that simple interest gives a higher return when the period is less than a year.
2. In pre-calculator days the calculation of $500(1.035)^{173/3}$ would require some work. Log tables, if available, could give the answer quickly but if only interest tables were available, you might have to write the product as $500(1.035)^{50}(1.035)^{7}(1.035)^{2/3}$. The values of $(1.035)^{50}$ and $(1.035)^{7}$ could be found in the interest tables, in particular in the $n = 50$ and $n = 7$ rows of the $i = 3\frac{1}{2}\%$ table. There is no $n = 57$ row of most interest tables, which is why $(1.035)^{57}$ would have to be broken up into two parts. The term $(1.035)^{2/3}$ presents a special problem. Usually only integral values of n are given in the interest tables, along with common fractional values such as $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{12}$, but not $\frac{2}{3}$. One could work this out by observing that $(1.035)^{2/3} = [(1.035)^{1/12}]^{8}$, but otherwise log tables or a power series expansion would be required.

Example 2.2

Alice borrows 5000 from The Friendly Finance Company at a rate of interest of 18% per year convertible semiannually. Two years later she pays the company 3000. Three years after that she pays the company 2000. How much does she owe seven years after the loan is taken out?

Solution

We will use a time diagram to aid in our solution:

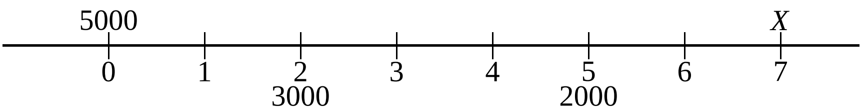

FIGURE 2.1

Let X be the amount still owing. In this type of problem, our goal is to obtain an equation of value which will yield the solution. To do that, all entries on the time diagram should be brought to the same point in time so an equation can be found. Any point in time can be chosen, but the most convenient one in this example is $t = 7$. The amount owing will equal the accumulated value at time 7 of the loan, minus the accumulated value at time 7 of the payments already made. Since the actual rate of interest is .09 effective per half-year, we have $X = 5000(1.09)^{14} - 3000(1.09)^{10} - 2000(1.09)^4 = 6783.38$. □

Example 2.3

Eric deposits 8000 in an account on January 1, 1995. On January 1, 1997, he deposits an additional 6000 in the account. On January 1, 2001, he withdraws 12,000 from the account. Assuming no further deposits or withdrawals are made, find the amount in Eric's account on January 1, 2004, if $i = .05$.

Solution

In this example, we see that withdrawals can be viewed as "negative deposits" in an equation of value.

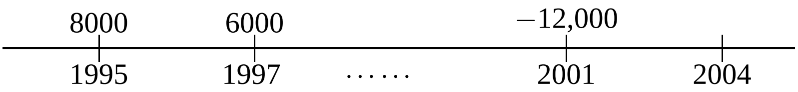

FIGURE 2.2

The resulting balance is

$$X = 8000(1.05)^9 + 6000(1.05)^7 - 12{,}000(1.05)^3 = 6961.73. \qquad \square$$

Example 2.4

John borrows 3000 from The Friendly Finance Company. Two years later he borrows another 4000. Two years after that he borrows an additional 5000. At what point in time would a single loan of 12,000 be equivalent if $i = .18$?

Solution

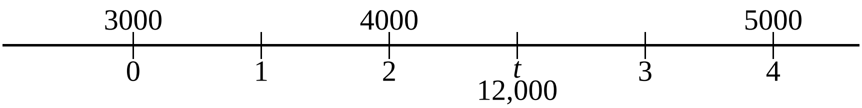

FIGURE 2.3

We let t be the number of years after the 3000 loan at which a single loan of 12,000 would be equivalent, and form the equation of value at time 0 as $12{,}000v^t = 3000 + 4000v^2 + 5000v^4$, where $v = \frac{1}{1.18}$. Then $v^t = \frac{3 + 4v^2 + 5v^4}{12}$. Taking logs of both sides of this equation we find $t = \frac{ln(3+4v^2+5v^4) - ln\,12}{ln\,v} = 2.11789$. $\square$

We remark that there is an approximate method of solving problems like Example 2.4, called the *method of equated time*, but we will not need to examine it here since there are no difficulties in obtaining an exact solution.

To conclude this section, we give a very simple example where the rate of interest is the unknown.

Example 2.5

Find the rate of interest such that an amount of money will triple itself over 15 years.

Solution

Let i be the required effective rate of interest. We have $(1 + i)^{15} = 3$, so that $i = 3^{1/15} - 1 = .07599$. $\square$

2.2 UNKNOWN RATE OF INTEREST

When the rate of interest is the unknown in an equation of value, complications often arise. To illustrate this, consider the following example.

Example 2.6

Joan deposits 2000 in her bank account on January 1, 1995, and then deposits 3000 on January 1, 1998. If there are no other deposits or withdrawals and the amount of money in the account on January 1, 2000 is 7100, find the effective rate of interest she earns.

Solution

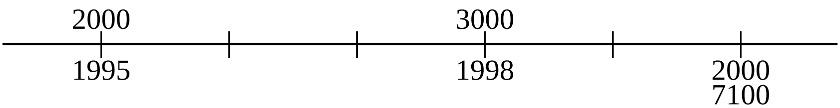

FIGURE 2.4

$2000(1+i)^5 + 3000(1+i)^2 = 7100$ is the equation of value on January 1, 2000. Now we have a problem. This equation is a fifth degree polynomial in i, and there is no exact formula for finding its solution. Most students will have a subroutine available on their calculators which will enable them to approximate the answer with a high degree of accuracy, and we encourage this approach. To show how these approximations are actually obtained, we will work out this example numerically.

Let $f(i) = 2000(1+i)^5 + 3000(1+i)^2 - 7100$. We wish to find two values for i, i_1 and i_2, such that $f(i_1) < 0$ and $f(i_2) > 0$, where i_1 and i_2 are close together. Then linear interpolation will be used to approximate a value i_0 such that $f(i_0) = 0$. To find i_1 and i_2, we use trial and error, aided by the fact that $f(i)$ is an increasing function. We eventually obtain $f(.11) = -33.58$ and $f(.12) = 187.88$.

Linear interpolation assumes that the function is a straight line between .11 and .12. The total change in the value of the function between $i = .11$ and $i = .12$ is $187.88 - (-33.58) = 221.46$. The amount of this change that occurs between .11 and a value i_0 such that $f(i_0) = 0$ is $0 - (-33.58) = 33.58$. Hence the fraction of the change occurring between .11 and i_0 is $\frac{33.58}{221.46} = .15163$, and i_0 must be that fraction of the distance between .11 and .12. This reasoning leads us to the conclusion that $i_0 = .11 + (.15163)(.01) = .1115163$, or $i_0 = .11152$ to five decimal places. □

Example 2.7

Obtain a more exact answer to Example 2.6.

Solution

To improve on the answer, we will start with values $i_1$and i_2 such that $f(i_1) < 0$ and $f(i_2) > 0$, where i_1 and i_2 are closer together than they were in the solution to Example 2.6. For instance, using $i_1 = .111$ and $i_2 = .112$, we find $f(.111) = -11.71$ and $f(.112) = 10.22$. Using these values, we obtain

$$i_0 = .111 + \left[\frac{11.71}{10.22-(-11.71)}\right](.001) = .11153. \qquad \square$$

We remark that standard calculator techniques give $i_0 = .11153$ as the correct answer (to five decimal places).

2.3 TIME-WEIGHTED RATE OF RETURN

The rate of interest calculated in Section 2.2 is often called the *dollar-weighted* rate of investment return. A very different procedure is used to calculate the *time-weighted* rate of investment return, and that is what we will consider here. We remark before starting that in this section the compound interest assumption is no longer being made.

To calculate the time-weighted rate of return, it is necessary to know the accumulated value of an investment fund just before each deposit or withdrawal occurs. Let B_0 be the initial balance in a fund, B_n the final balance, $B_1, \ldots, B_{n-1}$ the intermediate values just preceding deposits or withdrawals, and $W_1, \ldots, W_{n-1}$ the amount of each deposit or withdrawal, where $W_i > 0$ for deposits and $W_i < 0$ for withdrawals. Let $W_0 = 0$. Then

$$i_t = \frac{B_t}{B_{t-1} + W_{t-1}} - 1 \tag{2.1}$$

represents the rate of interest earned in the time period between balances B_{t-1} and B_t. The time-weighted rate of return is then defined by

$$i = (1 + i_1)(1 + i_2)\cdots(1 + i_n) - 1. \tag{2.2}$$

Example 2.8

On January 1, 1999, Graham's stock portfolio is worth 500,000. On April 30, 1999, the value has increased to 525,000. At that point, Graham adds 50,000 worth of stock to his portfolio. Six months later, the value has dropped to 560,000, and Graham sells 100,000 worth of stock. On December 31, 1999, the portfolio is again worth 500,000. Find the time-weighted rate of return for Graham's portfolio during 1999.

Solution

The accumulation rate from January 1 to April 30 is given by the factor $1+i_1 = \frac{525{,}000}{500{,}000} = 1.05$. Immediately after the April 30 stock purchase, the portfolio is worth 575,000. Hence the accumulation rate from May 1 to October 31 is $\frac{560{,}000}{575{,}000} = .97391$. Finally, the accumulation rate in the last two months of the year is $\frac{500{,}000}{460{,}000} = 1.08696$. The time-weighted rate of return for the year is found from the interval accumulation factors as $(1.05)(.97391)(1.08696)-1 = .11153$. □

Note in Example 2.8 that the value of the portfolio decreased during the period from May 1 to October 31, so we see that compound interest is clearly not operating here. Nevertheless, it is still possible to calculate a dollar-weighted rate of return by considering only deposits and withdrawals, and ignoring intermediate balances. Setting up the equation of value by accumulating all quantities to December 31, 1999, we obtain

$$500{,}000(1+i) + 50{,}000(1+i)^{2/3} - 100{,}000(1+i)^{1/6} = 500{,}000.$$

This could be solved by linear interpolation, as in Section 2.2, but an alternative approach to this type of problem is to assume simple interest for periods less than a year. We would then obtain

$$500{,}000(1+i) + 50{,}000(1+\tfrac{2}{3}\,i) - 100{,}000(1+\tfrac{1}{6}i) = 500{,}000.$$

Since this equation is linear in i, the result $i = \frac{300{,}000}{3{,}100{,}000} = .09677$ is easily obtained.

In Chapter 12 we will see how the theories of dollar-weighted and time-weighted rates of investment return are applied to pension funds.

EXERCISES

2.1 Equation of Value

2-1. Brenda deposits 7000 in a bank account. Three years later, she withdraws 5000. Two years after that, she withdraws an additional 3000. One year after that, she deposits an additional 4000. Assuming $i = .06$, and that no other deposits or withdrawals are made, how much is in Brenda's account ten years after the initial deposit is made?

2-2. Eileen borrows 2000 on January 1, 1997. On January 1, 1998, she borrows an additional 3000. On January 1, 2001, she repays 4000. Assuming $i = .13$, how much does she owe on January 1, 2005?

2-3. Boswell wishes to borrow a sum of money. In return, he is prepared to pay as follows: 200 after 1 year, 500 after 2 years, 500 after 3 years and 700 after 4 years. If $i = .12$, how much can he borrow?

2-4. Payments of 800, 500 and 700 are made at the ends of years 2, 3 and 6 respectively. Assuming $i = .13$, find the point at which a single payment of 2100 would be equivalent.

2-5. A vendor has two offers for a house: (i) 40,000 now and 40,000 two years hence, or (ii) 28,750 now, 23,750 in one year, and 27,500 two years hence. He makes the remark that one offer is "just as good" as the other. Find the two possible rates of interest which would make his remark correct.

2-6. (a) The present value of 2 payments of 1000 each, to be made at the end of n years and $n+4$ years, is 1250. If $i = .08$, find n.
(b) Repeat part (a) if the payments are made at the end of n years and $4n$ years.

2-7. In return for payments of 400 at the end of 3 years and 700 at the end of 8 years, a woman agrees to pay X at the end of 4 years and $2X$ at the end of 6 years. Find X if $i = .14$.

2-8. How long should 1000 be left to accumulate at $i = .12$ in order that it may amount to twice the accumulated value of another 1000 deposited at the same time at 8% effective?

2-9. Fund A accumulates at 9% effective and Fund B at 8% effective. At the end of 10 years, the total of the two funds is 52,000. At the end of 8 years, the amount in Fund B is three times that in Fund A. How much is in Fund A after 15 years?

2-10. John pays Henry 500 every March 15 from 1996 to 2000 inclusive. He also pays Henry 300 every June 15 from 1998 to 2001 inclusive. Assuming $i^{(4)} = .17$, find the value of these payments on (a) March 15, 2005; (b) March 15, 1999; (c) March 15, 1995.

2.2 Unknown Rate of Interest

2-11. A consumer purchasing a refrigerator is offered two payment plans:
Plan A: 150 down, 200 after 1 year, 250 after 2 years
Plan B: 87 down, 425 after 1 year, 50 after 2 years
Determine the range of interest rates for which Plan A is better for the consumer.

2-12. Find the effective rate of interest if payments of 300 at the present, 200 at the end of one year, and 100 at the end of two years accumulate to 800 at the end of three years.

2-13. Bernie borrows 5000 on January 1, 1995, and another 5000 on January 1, 1998. He repays 3000 on January 1, 1997, and then finishes repaying his loans by paying 10,000 on January 1, 2000. What effective annual rate of interest is Bernie being charged?

2-14. John buys a TV for 600 from Jean. John agrees to pay for the TV by making a cash down payment of 50, then paying 100 every four months for one year (i.e. three payments of 100), and finally making a single payment 16 months after the purchase (i.e. four months after the last payment of 100).
(a) Find the amount of the final payment if John is charged interest at an effective rate of 12% per year.
(b) Find the effective annual interest rate if John's final payment is 350.

2-15. A trust company pays 7% effective on deposits at the end of each year. At the end of every four years, a 5% bonus is paid on the balance at that time. Find the effective rate of interest earned by an investor if he leaves his money on deposit for (a) 3 years; (b) 4 years; (c) 5 years.

2-16. The present value of a series of payments of 1 at the end of every 3 years forever is equal to $\frac{125}{91}$. Find the effective rate of interest per year.

2.3 Time-Weighted Rate of Return

2-17. Emily's trust fund has a value of 100,000 on January 1, 1997. On April 1, 1997, 10,000 is withdrawn from the fund, and immediately after this withdrawal the fund has a value of 95,000. On January 1, 1998, the fund's value is 115,000.

(a) Find the time-weighted rate of investment return for this fund during 1997.

(b) Find the dollar-weighted annual rate of investment return for Emily's fund, assuming simple interest.

(c) Find the rate of return for Emily's fund using simple interest, and assuming a uniform distribution throughout the year of all deposits and withdrawals.

2-18. Assume in Question 17 that, in addition to the information given, there is also a 5000 deposit to the fund on July 1, 1997.

(a) Find the dollar-weighted annual rate of investment return for the fund, assuming simple interest.

(b) Find the rate of return for Emily's fund using simple interest and assuming a uniform distribution throughout the year of all deposits and withdrawals.

(c) Is it possible to calculate the time-weighted rate of return? If not, why not?

2-19. Let A be the balance in a fund on January 1, 1999, B the balance on June 30, 1999, and C the balance on December 31, 1999.

(a) If there are no deposits or withdrawals, show that the dollar-weighted and time-weighted rates of return for 1999 are both equal to $\frac{C-A}{A}$.

(b) If there was a single deposit of W immediately *after* the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted rates of return for 1999. (Assume simple interest for periods of less than a year.)

(c) If there was a single deposit of W immediately *before* the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted rates of return for 1999. (Assume simple interest for periods of less than a year.)

(d) Give a verbal explanation for the fact that the dollar-weighted rates of return in parts (b) and (c) are equal.

(e) Show that the time-weighted rate of return in part (b) is larger than the time-weighted rate of return in part (c).

CHAPTER THREE
ANNUITIES

3.1 ARITHMETIC AND GEOMETRIC SEQUENCES

Before beginning the study of annuities, we will briefly review some basic facts about arithmetic and geometric sequences. The formulae we develop for geometric sequences will be used a number of times later in the chapter.

Recall that an arithmetic sequence is a sequence of numbers $X_1, X_2, \ldots$ where the difference between consecutive terms is constant. For example, the sequence 4, 7, 10, 13, ... is an arithmetic sequence with constant difference 3 (assuming the apparent pattern continues). The sequence 5, 1, -3, -7, ... is also arithmetic, having common difference -4.

Any arithmetic sequence is determined by its first term and its common difference; if the first term is a and the common difference d, the sequence is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots. \tag{3.1}$$

There are two important formulae about arithmetic sequences which we would like to develop.

Theorem 3.1

Consider an arithmetic sequence with first term a and common difference d.

(a) The n^{th} term of this sequence is

$$a+(n-1)d.$$

(b) The sum of the first n terms of this sequence is given by the formula

$$\frac{n}{2}[2a+(n-1)d].$$

Proof

(a) Informally, we see from the pattern a, $a+d$, $a+2d$, $a+3d$, $\cdots$ that the n^{th} term will be $a+(n-1)d$. A formal proof can be given using mathematical induction. We will leave this as an exercise for the reader.

(b) Let S_n denote the sum of the first n terms. Using the result of part (a), we have

$$S_n = a + (a+d) + (a+2d) + \cdots + [a+(n-1)d]. \tag{3.2a}$$

Writing the terms on the right hand side of (3.2a) in the reverse order, we have

$$S_n = [a+(n-1)d] + [a+(n-2)d] + \cdots + (a+d) + a. \tag{3.2b}$$

Adding together (3.2a) and (3.2b) by combining first terms together, then second terms together, and so on, we obtain

$$\begin{aligned} 2S_n \;=\; & [a + a + (n-1)d] + [(a+d) + a + (n-2)d] \\ & + \cdots + [a + (n-1)d + a]. \end{aligned} \tag{3.2c}$$

There are n equal terms on the right side of (3.2c), so we have $2S_n = n[2a+(n-1)d]$, or

$$S_n = \frac{n}{2}[2a+(n-1)d], \tag{3.2d}$$

as required. As an exercise, the reader should give an alternate derivation of (3.2d) using mathematical induction. ∇

Example 3.1

Find the 32^{nd} term and the sum of the first 18 terms of the arithmetic sequence 5, 9, 13, 17, ...

Solution

We have $a = 5$ and $d = 4$. Hence, the 32^{nd} term is $5 + (31)(4) = 129$. Using (3.2d) the sum of 18 terms is $\frac{18}{2}[10+(17)4] = 702$. $\square$

Now we will consider geometric sequences. Recall that a geometric sequence is a sequence of numbers X_1, X_2, $\ldots$ where each consecutive term is obtained from the previous term by multiplying by a fixed number. For example, the sequence 2, 6, 18, 54, ... is geometric with common ratio 3, and the sequence 5, $-\frac{5}{2}$, $\frac{5}{4}$, $-\frac{5}{8}$, ... is geometric with common ratio $-\frac{1}{2}$. The general geometric sequence with first term a and common ratio r will be

$$a,\ ar,\ ar^2,\ ar^3, \ldots. \tag{3.3}$$

Theorem 3.2

Consider a geometric sequence with first term a and common ratio r.

(a) The n^{th} term of the sequence is ar^{n-1}

(b) The sum of the first n terms of the sequence is $\dfrac{a(1-r^n)}{1-r}$.

Proof

(a) As before, the pattern $a, ar, ar^2, \ldots$ leads us to believe that ar^{n-1} is the correct term, but a formal proof requires mathematical induction. We will leave this proof as an exercise for the reader.

(b) Let S_n be the sum of the first n terms. Using the result of part (a), we have

$$S_n = a + ar + \cdots + ar^{n-1}. \tag{3.4a}$$

Multiplying both sides by r, we obtain

$$rS_n = ar + ar^2 + \cdots + ar^n. \tag{3.4b}$$

Subtracting the second line from the first, we observe that many terms cancel, and we obtain

$$\begin{aligned} S_n - rS_n \ =\ & a + ar - ar + ar^2 - ar^2 \ + \ \cdots \\ & + ar^{n-1} - ar^{n-1} - ar^n = a - ar^n. \end{aligned} \tag{3.4c}$$

Thus we have $S_n(1-r) = a(1-r^n)$, or

$$S_n = \frac{a(1-r^n)}{1-r}, \tag{3.4d}$$

as required. ∇

Example 3.2

Find the 13^{th} term and the sum of the first 9 terms of the geometric sequence $48, -24, 12, -6, 3, -\frac{3}{2}, \ldots$.

Solution

We have $a = 48$ and $r = -\frac{1}{2}$. Using part (a) of Theorem 3.2, we find that the 13^{th} term is $48(-\frac{1}{2})^{12} = \frac{3}{256}$. Using (3.4d), the sum of the first 9 terms is $48\left[\frac{1-(-\frac{1}{2})^9}{1-(-\frac{1}{2})}\right] = 48\left[\frac{1+\frac{1}{512}}{\frac{3}{2}}\right] = \frac{513}{16}$. □

3.2 BASIC RESULTS

John borrows 1500 from a finance company and wishes to pay it back with equal annual payments at the end of each of the next ten years. If $i = .17$, what should his annual payment be?

Jacinta buys a house and takes out a 50,000 mortgage. If the mortgage rate is 13% convertible semiannually, what should her monthly payment be to pay off the mortgage in 20 years?

Eileen deposits 2000 in a bank account every year for 11 years. If $i = .06$, how much has she accumulated at the time of the last deposit?

All of these questions have one thing in common: they involve a series of payments made at regular intervals. Such a series of payments is called an *annuity*. In the three cases above, the payments are of equal amount, and that will be the case with all annuities studied in this section. Later, however, we will study more general annuities.

Annuities turn up in many different types of financial transactions. From the point of view of practical applications, a complete understanding of annuities is an absolute must!

We shall start by considering an annuity under which payments of 1 are made at the end of each period for n periods. Sometimes a period will be one year, as with John's loan above, but other periods are certainly possible. It will be assumed throughout that, as with John's loan, the interest period and the payment period are equal. When this is not the case, as with Jacinta's mortgage, for example, we will first find the equivalent rate of interest per payment period and then proceed with our solution.

Level payments of an amount other than 1 can be handled by multiplying by the amount of the payment, as we shall see in the examples.

FIGURE 3.1

A time diagram showing n payments of 1 is given in Figure 3.1. The present value of this annuity at time 0 is denoted by $a_{\overline{n}|}$. The accumulated value of this annuity at time n is denoted by $s_{\overline{n}|}$.

We shall now derive a formula for $a_{\overline{n}|}$. Taking the value at time 0 of each of the payments in turn, we obtain

$$a_{\overline{n}|} = v + v^2 + v^3 + \cdots + v^n. \tag{3.5}$$

This is the sum of n terms of a geometric sequence with $a = v$ and $r = v$. Using Formula (3.4d) developed in Section 3.1, we obtain

$$\begin{aligned} a_{\overline{n}|} &= \frac{v(1-v^n)}{1-v} \\ &= \frac{1-v^n}{\frac{1}{v}-1} \\ &= \frac{1-v^n}{1+i-1} \\ &= \frac{1-v^n}{i}. \end{aligned} \tag{3.6}$$

Formula (3.6) is crucial, and will be used frequently throughout the rest of the text.

It is easy now to get a formula for $s_{\overline{n}|}$. Since $s_{\overline{n}|}$ is the value of the same annuity n years after $a_{\overline{n}|}$ has been calculated, it follows that

$$\begin{aligned} s_{\overline{n}|} &= a_{\overline{n}|}(1+i)^n \\ &= \left[\frac{1-v^n}{i}\right](1+i)^n \\ &= \frac{(1+i)^n - v^n(1+i)^n}{i} \\ &= \frac{(1+i)^n - 1}{i}. \end{aligned} \tag{3.7}$$

Let us immediately proceed to some practical examples.

Example 3.3

Find John's payment in the problem stated in the first paragraph of this section.

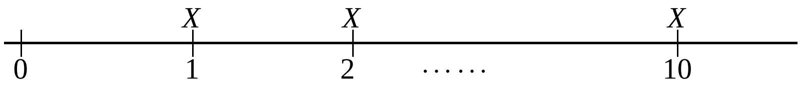

FIGURE 3.2

Solution

Let the payment be X. Since the present value of 10 payments of 1 is $a_{\overline{10|}}$, the present value of 10 payments of X will be $X \cdot a_{\overline{10|}}$. Thus we have

$$1500 = X \cdot a_{\overline{10|}}. \text{ Then } X = \frac{1500}{a_{\overline{10|}}} = \frac{1500}{\frac{1-v^{10}}{i}} = \frac{1500(.17)}{1-(\frac{1}{1.17})^{10}} = 321.98.$$

□

Example 3.4

Find the accumulated value in Eileen's bank account in the problem stated in the third paragraph of this section.

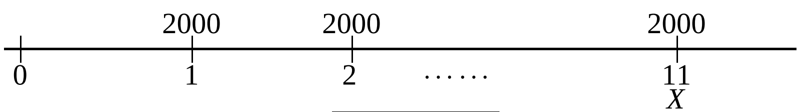

FIGURE 3.3

Solution

Since each deposit is 2000, the accumulated value will be given directly by $X = 2000s_{\overline{11|}} = 2000\left[\frac{(1.06)^{11}-1}{.06}\right] = 29{,}943.29.$ □

Example 3.5

Find Jacinta's mortgage payment in the problem stated in the second paragraph of this section.

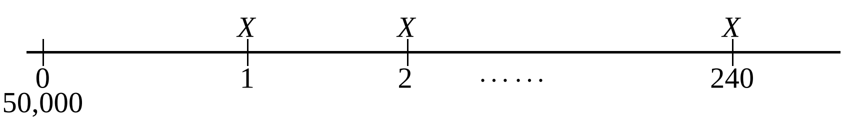

FIGURE 3.4

Solution

As mentioned earlier, we first have to find the effective monthly rate of interest equivalent to 13% convertible semiannually. This is because our formulae for $a_{\overline{n}|}$ and $s_{\overline{n}|}$ are based on the assumption that the interest period and payment period are the same. Letting this monthly rate be j, we have $1 + j = \left[1 + \frac{.13}{2}\right]^{1/6}$. Now we let the mortgage payment be X. Note that there are 240 monthly payments in the 20-year term of the mortgage, so we have $X \cdot a_{\overline{240}|} = 50{,}000$ and $X = \dfrac{50{,}000j}{1 - v^{240}} = 573.77$. □

Example 3.6

Elroy takes out a loan of \$5000 to buy a car. No payments are due for the first 8 months, but beginning with the end of the 9^{th} month, he must make 60 equal monthly payments. If $i = .18$, find (a) the amount of each payment; (b) the amount of each payment if there is no payment-free period, (i.e., if the first payment is due in one month and the remaining 59 are made on a monthly basis thereafter).

Solution

(a) We first note that a monthly rate of interest j is required. Since $(1 + j)^{12} = 1.18$, we obtain $j = (1.18)^{1/12} - 1$. Let the amount of each payment be X.

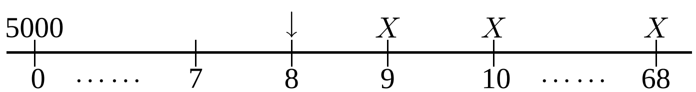

FIGURE 3.5

We now observe that this does not fit into the standard annuity pattern, since $X \cdot a_{\overline{60}|}$ will give us the value of the payments at month 8, one month before the first payment. The value of the loan at time 8 is $5{,}000(1+j)^8$, since it will accrue interest for eight months, even though no payments are required. Thus we have the equation of value $X \cdot a_{\overline{60}|} = 5000(1+j)^8$, so that $X = \dfrac{5000(1+j)^8}{a_{\overline{60}|}}$.

Evaluating $a_{\overline{60}|}$, $X = \dfrac{5000(1+j)^8 \cdot j}{1 - v^{60}} = 137.76$.

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